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The evolution of harmonic Indian musical drums: A mathematical perspective

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Abstract

We explain using mathematics how harmonic musical drums were discovered by Indian artisans and musicians more than 2000 years ago. To this end, we introduce a harmonic error function which measures the quality of the harmonic relationship and degeneracy of the first modes of vibration of a centrally symmetric loaded membrane. We explain that although the tabla configuration found by the ancient Indians is the most natural one, other configurations exist and some are harmonically superior to the classical one.

With the exception of the human voice, percussion instruments are probably the oldest known musical instruments. There are many different percussion instruments, but among the most popular are the drums whose vibrator is covered with one or two membranes. It is usually taken for granted that drums with circular membranes produce only non-harmonic vibrations, or in other words sounds without recognizable pitch. There are, however, surprising exceptions to this rule. From an ancient book dating back to circa 200 B.C. comes the legend of a magical drum given to a hermit by the Hindus god Brahmadatta [1]. According to this legend, the hermit could put his aggressors to flight by beating one side of the drum, and turn his enemies into trustworthy friends by beating the other side. This drum has the shape of a kettle and appears in paintings and carvings of many ancient temples in India, especially in the paintings of the Ajanta caves.

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Foremost among the Indian drums are the *tabla* and the *mrdanga*. The mrdanga has two membranes coupled by an enclosed air cavity and is played with the two hands. The tabla could be described as a mrdanga in two pieces with drumheads turned upwards. These two closed-shell instruments are called tabla and bhaya, or left-handed tabla, the tabla giving its name to the combination of the two instruments. A tabla pair is shown in Fig. 1.

The tabla and mrdanga differ essentially from the kettle and bass drums in having one membrane centrally loaded with a paste of iron-oxide, charcoal, starch, gum or other materials, that hardens but remains flexible. The mrdanga has only one loaded membrane, the other one being uniform. The process of loading a membrane is a long and delicate task. In the case of the mrdanga, for example, the patch results from the application of more than 100 successive fine layers [2]. The membrane of the bhaya is also loaded, but slightly off centre. In the bhaya usual mode of playing, the percussionist has the edge of his palm resting on the widest portion of the unloaded part of the membrane, and this constraint causes the nodal patterns to be quite symmetric. In the present paper, we shall be concerned only with centrally symmetric loaded membranes.

The tabla and mrdanga were first studied experimentally by a succession of scientists beginning with the Nobel laureate C.V. Raman [3–5]. Raman observed that the first four overtones of these drums form a sequence of natural harmonics with the fundamental, in contrast with the ordinary kettle and bass drums. Raman also concluded that the first nine modes of vibration having the lowest frequencies give a harmonic sequence of only five tones which means that some of these modes are degenerate, i.e. have approximately the same frequency. It is worth recalling that the theory of ordinary drumheads does not predict even approximate degeneracies of any of the modes or any harmonic relationships between the overtones.

The first mathematical model explaining Raman's observations was proposed by Ramakrishna and Sondhi [6]. In this model, the drumhead is represented as a circularly symmetric composite membrane. Using a graphical method, Ramakrishna and Sondhi found that if the loaded region extends to 40% of the membrane radius, then the configuration which optimizes mode degeneracies and the harmonic sequence of the overtones occurs when the loaded region is 9.76 times denser than the original membrane. Note that the value of 40% for the radius of the loaded region was chosen by Ramakrishna and Sondhi based on tabla configurations commonly found in practice.

From an archaeological standpoint, the following questions related to the evolution of the tabla and mrdanga arise: How did Indian artisans and musicians discover, more than 2000 years ago,

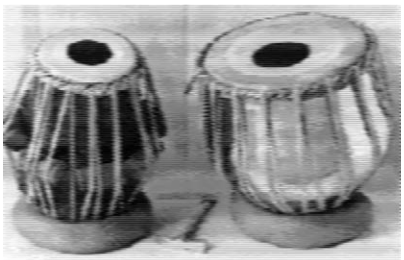

Fig. 1. A tabla pair.

an optimal configuration for these drums among an infinite number of possible configurations? Is it true that the probability of randomly obtaining such a configuration by loading a circular region of a given circularly uniform membrane would have been infinitesimally small? Could these drums have evolved in different configurations? We shall attempt to answer these intriguing questions mathematically by studying an error function which quantifies the quality of the harmonic sequences of loaded membranes which can be described using the Ramakrishna and Sondhi model.

The loaded tabla and mrdanga drumheads are modelled as circularly symmetric composite membranes. The density per unit area of the loaded region ($0 \leqslant r < a$) is represented by ρ_1, and ρ_2 represents the outer membrane ($a \leqslant r \leqslant b$) density. Here r represents the radial polar coordinate. The composite membrane is clamped at $r = b$ and under uniform tension τ per unit length. Only small-amplitude transverse oscillations with respect to the membrane equilibrium position are considered. We assume that the potential energy of the membrane is proportional to its change in surface area due to stretching. If $u_1(r, \theta, t)$ and $u_2(r, \theta, t)$ describe the displacement of the loaded ($i = 1$) and outer ($i = 2$) regions, respectively, it is straightforward to show, using Hamilton's principle, that the displacement of the composite membrane is governed by the wave equations

$$\nabla^2 u_i = c_i^{-2} \partial_t^2 u_i, \quad i = 1, 2,$$

with the conditions

$$u_i(r, \theta, t) = u_i(r, \theta + 2m\pi, t), \quad m = 0, \pm 1, \pm 2, \ldots, \; i = 1, 2$$

$$u_1(a, \theta, t) = u_2(a, \theta, t),$$

$$\partial_t u_1(a, \theta, t) = \partial_t u_2(a, \theta, t),$$

$$u_2(b, \theta, t) = 0,$$

where θ is the azimuthal polar coordinate, t represents time and $c_i = \sqrt{\tau/\rho_i}$, $i = 1, 2$.

Using the classical method of separation of variables, Ramakrishna and Sondhi showed that the eigenvalue equation for the above model can be expressed as

$$\sigma \frac{J_{n-1}(\sigma k x)}{J_n(\sigma k x)} = \frac{J_{n-1}(kx) Y_n(x) - J_n(x) Y_{n-1}(kx)}{J_n(kx) Y_n(x) - J_n(x) Y_n(kx)}, \tag{1}$$

whose solutions $x = 2\pi b f / c_2$ are the dimensionless eigenvalues from which follow the eigenfrequencies f. The Bessel functions of the first and second kind, and of order $n = 0, 1, \ldots$, are represented in Eq. (1) by $J_n(x)$ and $Y_n(x)$, respectively. We also have

$$k = a/b, \quad \sigma = c_2/c_1 = \sqrt{\rho_1/\rho_2}.$$

For a given set of values σ, k and n, each solution x of Eq. (1) leads to one of the eigenfrequencies which make up the frequency spectrum of the nth mode of vibration. For each given mode n, Eq. (1) yields an infinite number of solutions or eigenfrequencies f_{nj}, $j = 1, 2, \ldots$,. Ramakrishna and Sondhi solved Eq. (1) graphically for values of $2 < \sigma < 5$, after having fixed $k = 0.4$. According to their observations, the value which optimizes the degeneracy of the modes and the harmonic sequence appears to be $\sigma = 3.125$ in this case. Their results are summarized in the first three columns of Table 1. In the first column of this table, Ψ_{nj} represents the first nine modes of

Table 1
Ratios of normal mode frequencies

Mode	Uniform membrane	Composite membrane	Composite membrane	Composite membrane
		$k = 0.4$, $\sigma = 3.125$	$k = 0.4$, $\sigma = 3.125$	$k = 0.38$, $\sigma = 2.9$
Ψ_{01}	1.00	1.00 (0.00%)	1.00 (0.00%)	1.00 (0.00%)
Ψ_{11}	1.59	1.94 (−3.00%)	1.94 (−3.09%)	1.96 (−2.11%)
Ψ_{02}	2.40	3.06 (2.00%)	3.06 (1.97%)	3.05 (1.73%)
Ψ_{21}	2.14	2.95 (−1.67%)	2.95 (−1.70%)	3.00 (−0.05%)
Ψ_{12}	2.92	4.10 (2.50%)	4.11 (2.87%)	4.02 (0.51%)
Ψ_{31}	2.65	3.97 (−0.75%)	3.97 (−0.81%)	4.05 (1.37%)
Ψ_{03}	3.60	4.83 (−3.40%)	4.88 (−2.49%)	5.99 (−0.23%)
Ψ_{13}	4.23	—	7.00 (0.05%)	5.14 (2.75%)
Ψ_{22}	3.50	5.15 (3.00%)	5.18 (3.60%)	4.95 (−0.99%)
Ψ_{41}	3.16	4.96 (−0.80%)	4.96 (−0.72%)	5.09 (1.82%)

vibration of a circular membrane. The second column gives the ratios of the frequencies, with respect to the lowest frequency, for a circular membrane of uniform density (included for the sake of comparison). The third column gives the ratios of the frequencies, with respect to the lowest frequency, for the composite membrane with $k = 0.4$ and $\sigma = 3.125$. The maximum departure from ideal degeneracy occurs for mode Ψ_{03} and is approximately 3.4% for this case. The third, fourth and fifth harmonics are degenerate with two, two and three degenerate modes, respectively. The last two columns of Table 1 result from our analysis and give the ratios of the frequencies, again with respect to the lowest frequency, in the cases where $k = 0.4$ and $\sigma = 3.125$ (for comparison with Ramakrishna and Sondhi) and $k = 0.38$ and $\sigma = 2.9$. We explain below how these last two configurations were analysed.

We shall now explore a much broader range of the parameters σ and k than the one studied by Ramakrishna and Sondhi to see if there are any other theoretical composite membrane configurations which lead to approximate harmonic sequences with approximately the same number of degeneracies. In order to do this, a means of evaluating the quality of the harmonic sequence is required. To achieve this, we borrow an approach used in the design of progressive addition lenses (PAL) in the ophthalmology industry [7]. In order to determine the adequacy of a PAL design, an error function is used to evaluate the deviation of the measured power of the lenses from the prescribed power. The value of the error function at certain locations on the lenses is used as a measure of the quality of the design. Certain areas of the lens may be given higher priority (or more weight) than others. Applying a similar idea to the harmonic sequence of a composite membrane, we define the error function

$$E(\sigma, k) = \sum_{h=2}^{5} \omega_h \sum_{d=1}^{D} \left\{ \frac{(x_{nj}/x_{01}) - h}{h} \right\}^2, \tag{2}$$

where h represents the harmonic and D is the number of degenerate modes. Also, x_{nj} represents the solution of Eq. (1) corresponding to the jth eigenfrequency of the nth mode of vibration and

ω_h denotes the weight factor associated with the frequency deviations of harmonic h. Although it is not explicitly written in Eq. (2), in order to simplify the writing, D, n and j are all functions of h, σ and k. The choice of the fifth harmonic as the upper bound in the summation is somewhat arbitrary but remains consistent with previous studies. The higher overtones are highly damped and their corresponding modes contain very little energy [6]. The weight factors ω_h are included in the error function so that more or less importance can be given to certain near harmonics. This reflects certain *tabla* playing techniques where, for example, the fundamental frequency is muted and the higher harmonics play a more important role.

To evaluate the error function (2) for certain values of σ and k, a numerical solver is first necessary to find the solutions $x_{nj}(\sigma, k)$ of Eq. (1) for $n = 0, 1, \ldots$ and $j = 1, 2, \ldots$. We have used the Newton root-finding method. Once these solutions $x_{nj}(\sigma, k)$ are calculated, only the solutions corresponding to single or degenerate modes of the first five frequency overtones which correspond to approximate harmonics are kept. A criterion is required in order to determine if a frequency corresponds to a degenerate mode. The results we shall present below were obtained by considering as a degeneracy a departure of at most 10% from an exact harmonic and weight factors $\omega_h = 1$, for $h = 2, \ldots, 5$. With the number of degeneracies per harmonic identified as well as the corresponding values of n and j, the error function can be easily evaluated.

To explore the σk parameter space for possible harmonic configurations of the composite membrane, we evaluated the error function over the range of values $1.1 < \sigma < 10$ and $0.15 < k < 1$ and plotted the contour lines shown in Fig. 2. Optimal configurations of the composite membrane correspond in this figure to the lowest values of the error function. For clearer contour plots in

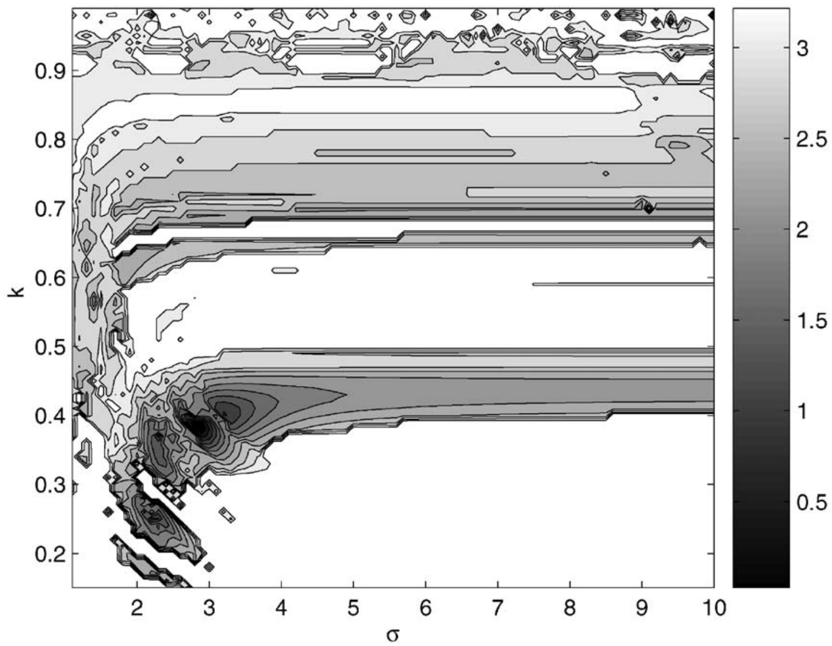

Fig. 2. Contour lines of error function (2) evaluated over the range $1.1 < \sigma < 10$ and $0.15 < k < 1$. The error function values plotted are $\ln(E(\sigma, k)) + 6.17$.

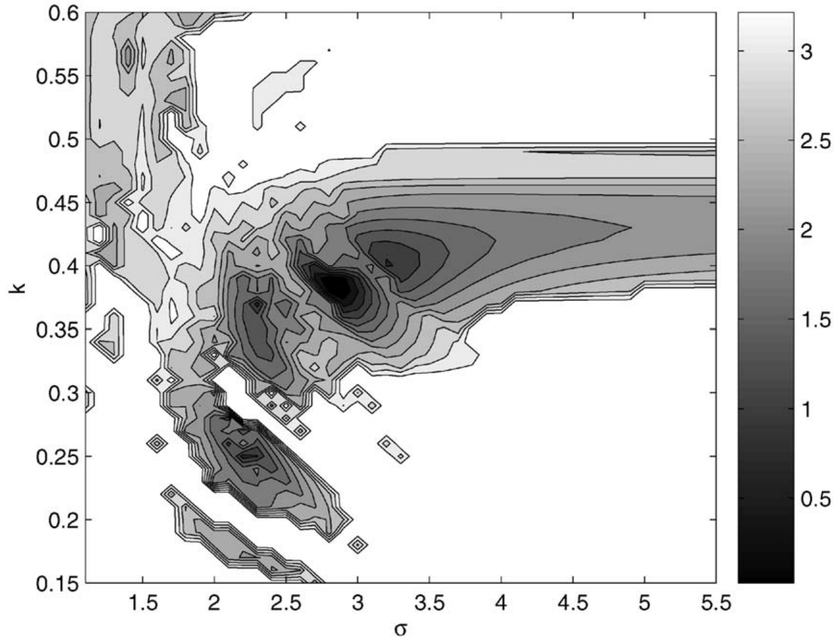

Fig. 3. A zoom of Fig. 1 over the range $1.1 < \sigma < 5.5$ and $0.15 < k < 0.6$. The error function values plotted are $\ln(E(\sigma, k)) + 6.17$.

Figs. 2 and 3, the logarithm of the error values has been translated by the amount 6.17 in order to avoid negative values. Because of the definition used for the error function (2), an undesirable solution could have a low error function value because it involves very few modes. These solutions are rejected if they contain seven or less modes. This criterion is based on Raman's analysis which contained nine modes. Upon examining Fig. 2, it becomes clear that the best solutions lie in a valley mostly localized along $k = 0.4$. Numerical tests performed with varying weight factors have shown that the location of this valley varies little and the error function values, and the location of the minima, vary slightly.

Fig. 3 shows a blown-up view of this valley which clearly contains several very good solutions. The solution identified by Ramakrishna and Sondhi ($k = 0.4$ and $\sigma = 3.125$) appears as a local minimum in Fig. 3 and has an error value of 0.0045. Using this value as reference, five other local minima in this valley can be found with error values of 0.0072 or less and can be identified in the plot as the darkest regions with closed concentric contour lines. Three of these solutions have error function values lower than the Ramakrishna and Sondhi solution. In particular, the most harmonic configuration of the composite membrane for the case where the degeneracies of the different harmonics have equal weights seems to occur for $k = 0.38$ and $\sigma = 2.9$ with an error value of 0.0021. This solution can still be considered as corresponding to the common tabla configuration. However, another solution with error value 0.0041 occurs for $k = 0.25$ and $\sigma = 2.2$ which represents a significant deviation from common tabla configurations which all seem to have a loaded region radius of $k = 0.4$. The error function minima which correspond to optimal tabla configurations are summarized in Table 2.

Table 2
The function $E(\sigma, k)$ for special values of σ and k

(σ, k)	$E(\sigma, k)$
(3.125,0.4)	0.0045
(2.9, 0.38)	0.0021
(2.3, 0.37)	0.0041
(2.3, 0.34)	0.0066
(2.2, 0.25)	0.0041
(2.1, 0.28)	0.0072

The contour plot in Fig. 3 suggests a mathematical explanation behind the evolution of the tabla at a loaded radius of $k = 0.4$ among an infinite number of possible configurations. The valley along $k = 0.4$ spans at least one order of magnitude of the square-root-density ratio σ. Therefore, the chances of stumbling onto a good configuration with an arbitrary σ are dramatically increased inside this valley which represents a significant surface area in Fig. 3. This would seem to explain the appearance of drums with a loaded region radius of $k = 0.4$.

The lack of repeatability in the application of the loaded region, due to limited technical means, may have actually contributed to the evolution of the Indian drums by accidentally allowing the configurations to randomly converge closer to the optimal solutions (moving left in the valley corresponding to $k = 0.4$ in Fig. 3). Remarkably, the mathematics underlying the vibrations of the tabla and mrdanga are such that the choice of $k = 0.4$ as the radius of the loaded region was a natural one with the most probability of being discovered. And, with this radius, the lack of ability to exactly reproduce the density ratio was probably a blessing which randomly lead to better sounding drums.

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